Multivariate algebraic generating functions: test problems

Torin Greenwood Stephen Melczer Tiadora Ruza Mark C. Wilson

November 22, 2021

This is a collection of (mostly multivariate) algebraic generating functions gleaned from the combinatorial literature, which we hope will be useful for future work. Most of them were found by Google Scholar keyword search. We intend to update the list regularly, and it will be available at the ACSV project website `https://acsvproject.org`.

1 Analysis of a random tree algorithm

Saunders [16] analyses a class of random recursive trees with deletion, by means of the GF

$$P(z,u) = \frac{q(1-u)zP^{\langle 0\rangle}(z) - u}{qz - u(1-puz)}$$

where

$$P^{\langle 0\rangle}(z) = \frac{2}{1 - 2qz + \sqrt{1-4pqz^2}}.$$

2 Bilateral Schröder paths

The number of bilateral Schröder paths (see `https://oeis.org/A063007`) from $(0,0)$ to (n,n) having k North steps is enumerated by

$$G(t,z) = \frac{1}{\sqrt{1-2z-4tz+z^2}.}$$

3 Orthogonal polynomials

For $\alpha \in \mathbb{R}$, the Gegenbauer polynomials $C_n^{(\alpha)}$ are orthogonal polynomials on $[-1,1]$ with respect to the weight function $(1-x^2)^{\alpha-1/2}$. The generating function is

$$F_\alpha(t,x) = \frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n\geq 0} C_n^{(\alpha)}(x)t^n = \sum_{n,k\geq 0} a_{nk}t^n x^k.$$

If α is a rational number (resp. integer) then F_α is algebraic (resp. rational). When $\alpha = 1$ we obtain the Chebyshev polynomials, and when $\alpha = 1/2$ we get Legendre polynomials.

Note: The Gegenbauer polynomials are a specialization of the Jacobi polynomials, which also yield algebraic GFs.

4 Parametrized univariate GF

Callan [7] describes the parametrized GF

$$f(x,a,b) = \frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx^2}.$$

Special cases of this yield shifted Catalan numbers, Motzkin numbers, little Schröder numbers. For each value of parameter the GF can be interpreted in terms of colored walks.

5 Edge flipping in the complete graph

Butler, Chung, Cummings, Graham [12] analyse the stationary distribution of a random walk on a state space of red/blue colorings of the complete graph, and obtain

$$f(x,y) = \frac{1 - x(1+y)}{\sqrt{1 - 2x(1+y) - x^2(1-y)^2}}.$$

6 Ising model integrals

Bostan *et al.* [10] consider many integrals associated with analysis of the n-particle contribution to the magnetic susceptibility of the Ising model. Among them we find [10, Eq. C.17]

$$\frac{1 - 2w + \sqrt{(1-2w)^2 - 4w^2t^2}}{2\sqrt{1-t^2}\sqrt{(1-2w)^2 - 4w^2t^2}} - \frac{1}{2}.$$

7 Generalization of Catalan numbers

Cossali [3] presents the following array. Let $g(n,m) = \frac{(2n+m)!}{m!n!(n+1)!}$. Then (bottom of page 5)

$$L(x,z) := \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} g(m,q) x^m z^q = \frac{(1-z) - \sqrt{(1-z)^2 - 4x}}{2x}.$$

Note that $L(x,0)$ encodes the Catalan numbers. Other fragments of this array encode triangular numbers and other known sequences.

8 A refinement of noncrossing partitions

Eu [1] presents this example. A partition of $\{1,2,\ldots,n\}$ is non-crossing if whenever $1 \le a < b < c < d \le n$ and whenever a, c are in the same block and b, d are in the same block, then a, b, c, and d are all in the same block. A partition is non-singleton if every block has size at least 2.

For linear representations of partitions, consider drawing the numbers $\{1,2,\ldots,n\}$ in a line and then drawing arcs connecting the numbers when they are in the same block. A block is called *visible* if it has no arcs from other blocks above it. So, in the partition $\{1,2,6\}, \{3,4,5\}, \{7,8\}, \{9,14\}\{10,13\}, \{11,12\}$, the blocks $\{1,2,6\}, \{7,8\}$, and $\{9,14\}$ are visible. Let $a(n,k)$ be the non-singleton non-crossing partitions of $[n]$ with k visible blocks. Then, by Theorem 60 (page 105),

$$A(z,u) := \sum_{u,k \ge 0} a_{n,k} u^k z^n = \frac{u(1 - z - 2uz^2 - \sqrt{1 - 2z - 3z^2})}{2(1 - u + zu + z^2u^2)}.$$

For circular representations of partitions, all blocks are visible. Thus, if $b_{n,k}$ is the number of circular non-crossing non-singleton partitions with k (visible) blocks, then by Proposition 63 (page 107),

$$B(x,u) := \sum_{n,k\geq 0} b_{n,k} u^k x^n = \frac{1+x-\sqrt{1-2x+x^2-4x^2u}}{2x(1+xu)}.$$

9 RNA secondary structures

Došlic, Svrtan & Veljan [4] discss RNA secondary structures. An RNA secondary structure can be viewed as a partial non-crossing pairing of the numbers $\{1,2,\ldots,n\}$. Let n be the *size* of the secondary structure. We sometimes add the condition that pairings must be distance ℓ apart, in which case we call ℓ the *rank*. Finally, let k be called the *order* of the secondary structure, corresponding to the total number of pairs in the structure. Fix a rank ℓ, and let $S_k^{(\ell)}(n)$ be the number of secondary structures of size n and order k. By Theorem 3.18 (page 80)

$$S_\ell(x,y) := \sum_{k,n\geq 0} S_k^{(\ell)}(n) y^k x^n = \frac{1}{2x^2y}\left[\Omega_\ell(x,y) - \sqrt{\Omega_\ell^2(x,y) - 4x^2y}\right]$$

where

$$\Omega_\ell(x,y) = (1-x)(1-y) - y\omega_\ell(x),$$
$$\omega_\ell(x) = x - 1 - x^2 \cdot \frac{1-x^\ell}{1-x}.$$

10 Dissections

Drmota [8, p. 376] presents a generating function $A(x,y)$ counting a certain type of graph ("dissections") with n nodes and m edges. It is given by the relation

$$A(x,y) = xy^2(1+A(x,y))^2 + xy(1+A(x,y)) \cdot A(x,y).$$

11 Ascents in Schröder paths

In Roitner's [15, Section 4.2] paper we find the bivariate GF that enumerates Schröder paths by semilength and number of ascents:

$$E(x,v) = \frac{1-xv-\sqrt{1-2x(v+2)+x^2(v-2)^2}}{2x(1+x(v-1))}.$$

12 Patterns in lattice paths

Asinowski and Banderier [13] discuss occurrences of very general patterns in lattice paths. Excursions, bridges, and meanders with marked patterns are encoded by algebraic generating functions. In Example 11 (page 7), they find the generating function encoding Dyck walks terminating on the x-axis with total steps marked by t, the number of occurrences of the pattern of steps $p_1 = udu$ marked by v_1, and the pattern $p_2 = dud$ marked by v_2:

$$B(t,v_1,v_2) = \sqrt{\frac{1+(1-v_1v_2)t^2+(1-v_1)(1-v_2)t^4}{1+(-3-v_1v_2)t^2+(1-v_1)(1-v_2)t^4}}.$$

13 Patterns in trees

Patterns within binary plane unlabelled trees where z and u track the size of the tree and number of occurrences of the pattern of length m, respectively, are given by [9, p. 680]

$$F(z,u) = \frac{1}{2z}\left(1 - \sqrt{1 - 4z - 4(u-1)z^{m+1}}\right).$$

14 Narayana numbers

These enumerate noncrossing partitions by set size and number of blocks, rooted ordered trees by edges and leaves, Dyck paths by semilength and number of peaks, etc (see `https://oeis.org/A001263`).

The bivariate sequence

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

is encoded in

$$F(x,t) = \frac{1 - x - xt - \sqrt{(1-x-xt)^2 - 4x^2t}}{2x}.$$

15 Assembly trees

Bona and Vince [11] define the concept of *assembly tree* of a graph and show that the generating function for the number of assembly trees of the complete bipartite graph K_{rs} is given by

$$\sum_{rs} a_{rs}x^r y^s = 1 - \sqrt{(1-x)^2 + (1-y)^2 - 1}.$$

16 New and old leaves

The Narayana numbers can be further refined by considering different types of leaves in a rooted ordered tree. [6] call a leaf of such a tree *old* if it is the leftmost child of its parent, and *young* otherwise. They enumerate such trees according to the number of old leaves, number of young leaves and number of edges, finding the algebraic equation

$$G(x,y,z) = 1 + \frac{z(G(x,y,z) - 1 + x)}{1 - z(G(x,y,z) - 1 + y)}.$$

17 Schröder trees by leaves and vertices

$$V(x,y) = xy + \frac{y(V(x,y))^2}{1 - V(x,y)}$$

18 Degree of symmetry of lattice paths

In [14, Lemma 2.1], Elizalde derives the generating function for bicolored Motzkin paths. Here, such a path starts and ends at the x-axis, never passes below the x-axis, and takes steps $U = (1,1), D = (1,-1)$, and two (colored) types of horizontal steps, $H_1 = (1,0)$ and $H_2 = (1.0)$. Let $a_{m,n}$ be the number of such paths with m total U or H_1 steps and n total D or H_2 steps. Then

$$M(x,y) := \sum_{m,n=0}^{\infty} a_{m,n}x^m y^n = \frac{1 - x - y - \sqrt{(1-x-y)^2 - 4xy}}{2xy}.$$

19 Bar graphs

Bousquet-Mélou & Rechnitzer [2] show that the bivariate GF $B(x, y)$ enumerating bar graphs by horizontal external edges and vertical external edges satisfies $B(0, 0) = 0$ and $P(x, y, B) = 0$, where $P(x, y, Y) = Y - xy - (x + y + xy)Y + xY^2$.

20 Planar maps and the quadratic method

Goulden & Jackson [5, Sec. 2.9.9] present the GF $M(x, y)$ enumerating rooted planar maps, where x marks edges in the outer face and y marks edges. They show the defining equation

$$(1 - x)(Y - 1) = x^2yY^2 + xy(h(y) - xY)$$

where $h(y) := M(1, y)$. The quadratic method is used to express $h(y)$ in the form $(1 - 4\beta)(1 - 3\beta)^{-2}$ where the algebraic univariate GF β is defined by $\beta(y) = y(1 - 3\beta)^{-1}$. They note that "no convenient expression has been obtained for $[x^ny^m]M(x, y)$".

References

[1] Sen-Peng Eu. "On the quadratic algebraic generating functions and combinatorial structures". PhD thesis. Ph. D. dissertation, National Taiwan Normal University, 2002.

[2] Mireille Bousquet-Mélou and Andrew Rechnitzer. "The site-perimeter of bargraphs". In: *Advances in Applied Mathematics* 31.1 (2003), pp. 86–112.

[3] GE Cossali. "A common generating function for Catalan numbers and other integer sequences". In: *Journal of Integer Sequences* 6.2 (2003), p. 3.

[4] Tomislav Došlić, Dragutin Svrtan, and Darko Veljan. "Enumerative aspects of secondary structures". In: *Discrete Mathematics* 285.1-3 (2004), pp. 67–82.

[5] Ian P. Goulden and David M. Jackson. *Combinatorial enumeration*. With a foreword by Gian-Carlo Rota, Reprint of the 1983 original. Mineola, NY: Dover Publications Inc., 2004, pp. xxvi+569. ISBN: 0-486-43597-0.

[6] William Y. C. Chen, Emeric Deutsch, and Sergi Elizalde. "Old and young leaves on plane trees". In: *European Journal of Combinatorics* 27 (2006), pp. 414–427.

[7] David Callan. "On generating functions involving the square root of a quadratic polynomial". In: *Journal of Integer Sequences* 10.2 (2007), p. 3.

[8] Michael Drmota. *Random trees: an interplay between combinatorics and probability*. Springer Science & Business Media, 2009.

[9] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, 2009. ISBN: 0521898064.

[10] Alin Bostan et al. "Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity". In: *arXiv preprint arXiv:1211.6031* (2012).

[11] Miklós Bóna and Andrew Vince. "The number of ways to assemble a graph". In: *2013 Proceedings of the Tenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*. SIAM. 2013, pp. 8–17.

[12] Steve Butler et al. "Edge flipping in the complete graph". In: *Advances in Applied Mathematics* 69 (2015), pp. 46–64.

[13] Andrei Asinowski and Cyril Banderier. "On Lattice Paths with Marked Patterns: Generating Functions and Multivariate Gaussian Distribution". In: *31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2020)*. Ed. by Michael Drmota and Clemens Heuberger. Vol. 159. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020, 1:1–1:16. ISBN: 978-3-95977-147-4. DOI: 10.4230/LIPIcs.AofA.2020.1. URL: https://drops.dagstuhl.de/opus/volltexte/2020/12031.

[14] Sergi Elizalde. "The degree of symmetry of lattice paths". In: *Ann. Combin.* (2021), pp. 1–35.

[15] Valerie Roitner. "The vectorial kernel method for walks with longer steps". In: *Quaestiones Mathematicae* (2021), pp. 1–19.

[16] Arnold T. Saunders. "A Class of Random Recursive Tree Algorithms with Deletion". In: *Algorithmica* 83.11 (2021), pp. 3363–3378.